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Occurrence, Repetition and Matching of Patterns in the Low-temperature Ising Model

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We continue our study of the exponential law for occurrences and returns of patterns in the context of Gibbsian random fields. For the low-temperature plus-phase of the Ising model, we prove exponential laws with error bounds for occurrence, return, waiting and matching times. Moreover we obtain a Poisson law for the number of occurrences of large cylindrical events and a Gumbel law for the maximal overlap between two independent copies. As a by-product, we derive precise fluctuation results for the logarithm of waiting and return times. The main technical tool we use, in order to control mixing, is disagreement percolation.

KEY WORDS: Disagreement percolation;exponential law; Poisson law; Gumbel law; large deviations.

1. INTRODUCTION

The study of occurrence and return times for highly mixing random fields has been initiated by Wyner, see ref. 16. In the context of stationary *processes*, there is a vast literature on exponential laws with error bounds for α, φ, ψ-mixing processes, see e.g. ref. 3. for a recent overview. In the last four years, very precise results were obtained by Abadi.$^{(2)}$ The advantage of his approach is that it gives sharp bounds on the error of the exponential approximation and it holds for *all* cylindrical events. Moreover, it can be generalized to a broad class of random fields, see ref. 4 for the case of Gibbsian random fields in the Dobrushin uniqueness regime (high temperature).

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Low-temperature Gibbsian random fields do not share the mixing property of the Dobrushin uniqueness regime, i.e. they are not (non-uniformly) φ-mixing. So far, no results on exponential laws have been proved in this context. To study these questions for Gibbsian random fields at low temperature, the Ising model is a natural candidate to begin with. The typical picture of the low-temperature plus-phase of this model is a sea of plus spins with exponentially damped islands of minus spins. Therefore decay of correlations of local observables can be estimated using the technique of disagreement percolation as initiated in ref. 5 and further exploited in ref. 6.

In this paper we prove the exponential law with error bounds for occurrences and returns of cylindrical events for the low-temperature plus-phase of the Ising model. As an application we also obtain the exponential law with error bounds for waiting and matching times. These results can then be further exploited to obtain a Poisson law for the number of occurrences of cylindrical events (the Poisson law for the number of large contour has been obtained in ref. 9 in the limit of zero temperature). We also derive a 'Gumbel law' for the maximal overlap (in the spirit of ref. 13) between two independent copies of the low-temperature Ising model. Other applications are strong approximations and large deviation estimates of the logarithm of waiting and return times. Our results are based upon disagreement percolation estimates and are not limited to the Ising model only. However in this paper we restrict to this example for the sake of simplicity.

The paper is organized as follows. In Section 2 we introduce basic notations, define occurrence and return times, and collect the mixing results at low temperature based on disagreement percolation. In Section 3 we state our results. Section 4 is devoted to proofs.

2. NOTATIONS, DEFINITIONS

2.1. Configurations, Ising Model

We consider the low-temperature plus-phase of the Ising model on $\mathbb{Z}^d$, $d \geqslant 2$. This is a probability measure $\mathbb{P}^+_\beta$ on lattice spin configurations $\sigma \in \Omega = \{+,-\}^{\mathbb{Z}^d}$, defined as the weak limit as $V \uparrow \mathbb{Z}^d$ of the following finite volume measures:

$$\mathbb{P}^+_{V,\beta}(\sigma_V) = \exp\left(-\beta \sum_{<xy>\in V} \sigma_x \sigma_y - \beta \sum_{\substack{<xy>: \\ x\in\partial V,\ y\notin V}} \sigma_x \right) \Big/ Z^+_{V,\beta} \tag{2.1}$$

where $Z^{+}_{V,\beta}$ is the partition function. In (2.1) $<xy>$ denotes nearest neighbor bonds and ∂V the inner boundary, i.e., the set of those $x\in V$ having at least one neighbor $y\notin V$. For the existence of the limit $V\uparrow\mathbb{Z}^d$ of $\mathbb{P}^{+}_{V,\beta}$, see e.g. ref. 11.

For $\eta\in\Omega$, $V\subseteq\mathbb{Z}^d$ we denote by $\mathbb{P}^{\eta}_{V,\beta}$ the corresponding finite volume measure with boundary condition η:

$$\mathbb{P}^{\eta}_{V,\beta}(\sigma_V)=\exp\left(-\beta\sum_{<xy>\in V}\sigma_x\sigma_y-\beta\sum_{\substack{<xy>:\\ x\in\partial V,\ y\notin V}}\sigma_x\eta_y\right)\Big/Z^{\eta}_{V,\beta}.$$

Later on, we shall omit the indices $\beta,+$ (in $\mathbb{P}^{+}_{\beta}$) referring to the inverse temperature and plus boundary condition respectively. We will choose $\beta>\beta_0>\beta_c$, i.e., temperature below the transition point, such that a certain mixing condition, defined in detail below, is satisfied.

Let $V_n\uparrow\mathbb{Z}^d_+$ be an increasing sequence of sets such that

$$\lim_{n\to\infty}\frac{|\partial V_n|}{|V_n|}=0.$$

In view of a later application to large deviation estimates, we need the following pressure function $q\mapsto P(q\beta)$, $q\in\mathbb{R}$:

$$P(q\beta)=\lim_{n\to\infty}\frac{1}{|V_n|}\log\sum_{\sigma_{V_n}\in\{+,-\}^{V_n}}\exp\left(-q\beta\sum_{<xy>\in V_n}\sigma_x\sigma_y\right). \qquad (2.2)$$

(See ref. 11 for the existence of $P(q\beta)$.)

2.2. Patterns, Occurrence, Repetition and Matching Times

A pattern supported on a set $V\subseteq\mathbb{Z}^d$ is a configuration $\sigma_V\in\{+,-\}^V$. Patterns will be denoted by A. We will identify A with its cylinder, i.e., with the set $\{\sigma\in\Omega:\sigma_V=A\}$, so that it makes sense to write e.g. $\sigma\in A$. For $x\in\mathbb{Z}^d$, θ_x denotes the shift over x. For a pattern A supported on V, $\theta_x A$ denotes the pattern supported on $V+x$ defined by $\theta_x A(y+x)=A(y)$, $y\in V$. We observe that for any Gibbs measure, so in particular in our context, we have the uniform estimate

$$\mathbb{P}(\sigma_V=A)\leqslant e^{-\delta|V|} \qquad (2.3)$$

for some $\delta > 0$ and all patterns A.

If A is a pattern supported on V, and $W \subseteq \mathbb{Z}^d$ then we denote by $(A \prec W)$ the event that there exists $x \in \mathbb{Z}^d$ such that $V + x \subseteq W$ and such that $\sigma_{V+x} = \theta_x A$. In words this means that the pattern A appears in the set W.

Let $\mathbb{V} = (V_n)$ where $V_n \uparrow \mathbb{Z}_+^d$, is such that $\lim_{n\to\infty} \frac{|\partial V_n|}{|V_n|} = 0$, and A_n a pattern supported on V_n. We define

$$\mathbf{T}_{A_n}^{\mathbb{V}} = \min\{|V_k| : A_n \prec V_k\}.$$

In words, this is volume of the first set V_k in which we can see the pattern A_n.

For $n \in \mathbb{N}$ let $\mathcal{C}_n$ be $[0,n]^d \cap \mathbb{Z}^d$. We denote for $x \in \mathbb{Z}^d$: $C(x,n) = \mathcal{C}_n + x$. For $x, y \in \mathbb{Z}^d$: $|x-y| = \max_{i=1}^d |x_i - y_i|$, and for subsets $A, B \subseteq \mathbb{Z}^d$: $d(A,B) = \min_{x\in A, y\in B} |x-y|$.

For $\sigma \in \Omega$, A a pattern supported on V, $W \supset V$, we define the number of occurrences of A in W:

$$N(A, W, \sigma) = \sum_{x\in W : V+x \subseteq W} I(\sigma_{V+x} = \theta_x A).$$

For a sequence $V_n \uparrow \mathbb{Z}_+^d$, the return time is defined as follows:

$$\mathbf{R}_{\sigma_{V_n}}(\sigma) = \min\{|V_k| : N(\sigma_{V_n}, V_k, \sigma) \geqslant 2\}.$$

Finally, for $\mathbb{V} = V_n \uparrow \mathbb{Z}_+^d$, and $\sigma, \eta \in \Omega$, we define the waiting time:

$$\mathbf{W}(V_n, \eta, \sigma) = \mathbf{T}_{\eta_{V_n}}^{\mathbb{V}}(\sigma).$$

We are interested in this quantity for σ distributed according to $\mathbb{P}$ and η distributed according to another ergodic (sometimes Gibbsian) probability measure $\mathbb{Q}$.

Finally, we consider 'matching times', in view of studying maximal overlap between two independent samples of $\mathbb{P}$. For $\sigma, \eta \in \Omega$,

$$\mathbf{M}(V_n, \sigma, \eta) = \min\{|V_k| : \exists x:\ V_n + x \subseteq V_k,\ \sigma_{V_n+x} = \eta_{V_n+x}\}.$$

In words, this is the minimal volume of a set of type V_k such that inside V_k, σ and η match on a set of the form $V_n + x$.

In the sequel we will omit the reference to the sequence V_n, in order not to overburden notation. In fact, proofs will be done for $V_n = \mathcal{C}_n = [0,n]^d \cap \mathbb{Z}^d$. The generalization to $\mathbb{V}$ is obvious provided that the following two (sufficient) conditions are fulfilled:

1. $\lim_{n\to\infty} \frac{|\partial V_n|}{|V_n|} = 0$;

2. There exists $c > 0$ such that, for all x with $|x| \geqslant 1$, $|(V_n + x) \Delta V_n| \geqslant cn$.

2.3. Mixing at Low Temperatures

In ref. 4 we derived exponential laws for hitting and return times under a mixing condition of the type

$$\sup_{\sigma,\eta,\xi} |\mathbb{P}_V^{\eta}(\sigma_W) - \mathbb{P}_V^{\xi}(\sigma_W)| \leqslant |W| \exp(-cd(V^c, W)) \tag{2.4}$$

usually called 'non-uniform exponential φ-mixing'. This condition is of course not satisfied at low temperatures since boundary conditions continue to have influence. Take e.g. $W = \{0\}$, $\eta \equiv +$, $\xi \equiv -$, then for $\beta > \beta_c$:

$$\lim_{V \uparrow \mathbb{Z}^d} \mathbb{P}_V^{\eta}(\sigma_0 = +) - \mathbb{P}_V^{\xi}(\sigma_0 = +) = m_\beta^+ > 0$$

where $0 < m_\beta^+ = \int \sigma_0 d\mathbb{P}(\sigma)$ is the magnetization. This clearly contradicts (2.4). However, for local functions f, g we do have an estimate like

$$\left| \int f \ \theta_x g \ d\mathbb{P} - \int f d\mathbb{P} \int g d\mathbb{P} \right| \leqslant C(f, g) \ e^{-c(\beta)|x|}.$$

The intuition here is that there can only be correlation between two functions if the clusters containing their dependence sets are finite (i.e., not contained in the sea of pluses) and intersect. Since finite clusters are exponentially small (in diameter), we have exponential decay of correlations of local functions.

This idea is formalized in the context of 'disagreement percolation'. To introduce this concept, we define a path $\gamma = \{x_1, \dots, x_n\}$, i.e., a subset of $\mathbb{Z}^d$ such that x_i and x_{i-1} are neighbors for all $i = 1, \dots, n$.

More formally, for $W \subseteq V$ and η and $\xi \in \Omega$, we have the following inequality:

$$|\mathbb{P}_V^{\eta}(\sigma_W) - \mathbb{P}_V^{\xi}(\sigma_W)| \leqslant |\partial W| \ \mathbb{P}_V^{\eta} \otimes \mathbb{P}_V^{\xi}(W \leftrightsquigarrow \partial V). \tag{2.5}$$

Here $(W \leftrightsquigarrow \partial V)$ denotes the event of those couples $(\sigma_1, \sigma_2) \in \Omega_V \times \Omega_V$ where there is 'a path of disagreement' γ leading from W to the boundary of V such that $\sigma_1(x) \neq \sigma_2(x)$ for all $x \in \gamma$. Of course whether the probability of this event under the measure $\mathbb{P}_V^{\eta} \otimes \mathbb{P}_V^{\xi}$ will be small depends on the distance between V and W and on the chosen boundary conditions η, ξ. The estimate (2.5) as well as the ideas of disagreement percolation can be found in refs. 6 and 13.

On the top of inequality (2.5) we have the following estimate of ref. 7, see ref. 12:

$$\mathbb{P} \otimes \mathbb{P}(\partial W \not\leftrightarrow \partial V) \leqslant e^{-c(\beta) d(W, \partial V)} \tag{2.6}$$

as soon as $\beta > \beta_0 (> \beta_c)$, and where $c(\beta) \to \infty$ as $\beta \to \infty$.

In the rest of the paper we always work with $\beta > \beta_0$, so that we can apply (2.5), (2.6). We emphasize that the next results are in fact valid not only for the Ising model at low temperature but also for any Markovian random field for which the above disagreement percolation estimates hold.

3. RESULTS

3.1. Exponential Laws

Theorem 1 **(Occurrence times).** There exist $0 < \Lambda_1 \leqslant \Lambda_2 < \infty$, $c, c' > 0$, $0 < \kappa < c$, such that for all patterns $A = A_n$ supported on $\mathcal{C}_n$, there exists $\lambda_A \in [\Lambda_1, \Lambda_2]$ such that for all n and all $t < e^{\kappa n^d}$:

$$\left| \mathbb{P}\left(\mathbf{T}_A \geqslant \frac{t}{\lambda_A \mathbb{P}(A)} \right) - e^{-t} \right| \leqslant e^{-ct} e^{-c' n^d}. \tag{3.1}$$

For return times we have to restrict to 'good patterns', i.e., patterns which are not 'badly self-repeating' in the following sense:

Definition 1. A pattern A_n is called good if for any x with $|x| < n/2$, for the cylinders we have $A_n \cap \theta_x A_n = \varnothing$.

Good patterns have a return time at least $(n/2+1)^d$ and as we will see later that this property guarantees that the return time is actually of the order e^{cn^d}.

The following lemma is proved in ref. 4 for general Gibbsian random fields.

Lemma 1. Let $\mathcal{G}_n$ be the set of all good patterns. There exists $c > 0$ such that

$$\mathbb{P}(\mathcal{G}_n) \geqslant 1 - e^{-cn^d}.$$

We denote by $\mathbb{P}(\cdot | A)$ the measure $\mathbb{P}$ conditioned on the event $A \prec \mathcal{C}_n$.

Theorem 2 (Repetition time). There exist $0 < \Lambda_1 \leqslant \Lambda_2 < \infty$, $c, c' > 0$, $0 < \kappa < c$, such that for all *good* patterns $A = A_n$ supported on $\mathcal{C}_n$, there exists $\lambda_A \in [\Lambda_1, \Lambda_2]$ such that for all n and all $t < e^{\kappa n^d}$:

$$\left| \mathbb{P}\left(\mathbf{R}_A \geqslant \frac{t}{\lambda_A \mathbb{P}(A)} \;\middle|\; A \right) - e^{-t} \right| \leqslant e^{-ct} e^{-c' n^d}. \tag{3.2}$$

We have the following analogue of Theorem 1 for matching times.

Theorem 3 (Matching time). There exist $0 < \Lambda_1 \leqslant \Lambda_2 < \infty$, $c, c' > 0$, $0 < \kappa < c$, such that for all patterns $A = A_n$ supported on $\mathcal{C}_n$, there exists $\lambda_A \in [\Lambda_1, \Lambda_2]$ such that for all n and all $t < e^{\kappa n^d}$:

$$\left| \mathbb{P} \otimes \mathbb{P}\left((\sigma, \eta) \colon \mathbf{M}_n(\sigma, \eta) \geqslant \frac{t}{\lambda_n \mathbb{P} \otimes \mathbb{P}(\sigma_{\mathcal{C}_n} = \eta_{\mathcal{C}_n})} \right) - e^{-t} \right| \leqslant e^{-ct} e^{-c' n^d}. \tag{3.3}$$

3.2. Poisson Law

Let $A = A_n$ be any pattern supported on $\mathcal{C}_n$. For $t > 0$, let $C(t/\mathbb{P}(A))$ be the maximal cube of the form $C_k = [0, k]^d \cap \mathbb{Z}^d$ such that $|C_k| \leqslant t/\mathbb{P}(A)$. Observe that

$$\frac{|C(t/\mathbb{P}(A))|}{t/\mathbb{P}(A)} \to 1$$

as $n \to \infty$. Define

$$N_t^n(\sigma) = N(A_n, C(t/\mathbb{P}(A)), \sigma). \tag{3.4}$$

Then we have

Theorem 4. If σ is distributed according to $\mathbb{P}$, and A_n is a sequence of good patterns, then the processes $\{N_t^n/\lambda_{A_n} : t \geqslant 0\}$ converge to a mean one Poisson process $\{N_t : t \geqslant 0\}$ weakly on path space, where λ_{A_n} is the parameter of Theorem 1.

3.3. Gumbel Law

To formulate the Gumbel law for certain extremes, we need simply connected subsets G_n, $n \geqslant 1$, such that $|G_n| = n$ and $G_{n^d} = \mathcal{C}_n$. For instance, for $d = 2$, $G_1 = \{(0,0)\}$, $G_2 = \{(0,0), (1,0)\}$, $G_3 = \{(0,0), (1,0), (1,1)\}$, $G_4 = \{(0,0), (1,0), (1,1), (0,1)\}$, etc.

For $\eta \in \Omega$, define

$$\mathcal{M}_n(\eta,\sigma) = \max\{|G_k| : \exists x \in G_n \text{ with } G_k + x \subseteq G_n \text{ and } \eta_{G_k+x} = \sigma_{G_k+x}\} \tag{3.5}$$

In words this is the volume of the maximal subset of the type G_k on which η and σ agree. We have the following

Theorem 5. For any $\eta \in \Omega$, there exists a sequence $u_n \uparrow \infty$, and constants $\lambda, \lambda', \nu, \nu' \in (0,\infty)$ such that for all $x \in \mathbb{Z}$

$$\begin{aligned}
&\min\{e^{-\lambda' e^{-\nu' x}}, e^{-\lambda e^{-\nu x}}\} \leqslant \liminf_{n\to\infty} \mathbb{P} \otimes \mathbb{P}\,((\eta,\sigma)\colon \mathcal{M}_n(\eta,\sigma) \leqslant u_n + x) \\
&\leqslant \limsup_{n\to\infty} \mathbb{P} \otimes \mathbb{P}\,((\eta,\sigma)\colon M_n(\eta,\sigma) \leqslant u_n + x) \leqslant \max\{e^{-\lambda' e^{-\nu' x}}, e^{-\lambda e^{-\nu x}}\}.
\end{aligned} \tag{3.6}$$

The fact that in the Gumbel law we only have a lower and an upper bound is due to the discreteness of the $\mathbf{M}_n(\sigma,\eta)$. This situation can be compared to the study of the maximum of independent geometrically distributed random variables, see for instance ref. 10.

Remark 1. Notice that in Theorem 5 we study the maximal matching between two configurations on a specific sequence of supporting sets G_n. Since in the low-temperature plus-phase we have percolation of pluses, the same theorem would of course not hold for the cardinality of the maximal connected subset of $\mathcal{C}_n$ on which η and σ agree because the latter subset occupies a fraction of the volume of $\mathcal{C}_n$.

3.4. Fluctuations of Waiting, Return and Matching Times

We denote by $s(\mathbb{P})$ the entropy of $\mathbb{P}$ defined by

$$s(\mathbb{P}) = \lim_{n\to\infty} -\frac{1}{n^d} \sum_{A_n \in \{+,-\}^{\mathcal{C}_n}} \mathbb{P}(A_n) \log \mathbb{P}(A_n).$$

The next result (proved in Subsection 4.7) shows how the repetition of typical patterns allows to compute the entropy from a single 'typical' configuration.

Theorem 6. There exists $\epsilon_0 > 0$ such that for all $\epsilon > \epsilon_0$

$$-\epsilon \log n \leqslant \log \left[\mathbf{R}_{\sigma_{\mathcal{C}_n}}(\sigma) \ \mathbb{P}(\sigma_{\mathcal{C}_n}) \right] \leqslant \log \log n^{\epsilon} \quad \text{eventually } \mathbb{P}\text{-almost surely.} \tag{3.7}$$

In particular,

$$\lim_{n\to\infty} \frac{1}{n^d} \log \mathbf{R}_{\sigma_{\mathcal{C}_n}}(\sigma) = s(\mathbb{P}) \quad \mathbb{P}\text{-almost surely.} \tag{3.8}$$

Note that (3.8) is a particular case of the result by Ornstein and Weiss in ref. 15 where $\mathbb{P}$ is only assumed to be ergodic. Under our assumptions, we get the more precise result (3.7).

Remark 2. It follows immediately from (3.7) that the sequence $(\log \mathbf{R}_{\sigma_{\mathcal{C}_n}}(\sigma)/n^d)$ satisfies the central limit theorem if and only if $(-\log \mathbb{P}(\sigma_{\mathcal{C}_n})/n^d)$ does. However, in the low-temperature regime, we are not able to prove the central limit theorem for $(-\log \mathbb{P}(\sigma_{\mathcal{C}_n})/n^d)$.

Suppose that η is a configuration randomly chosen according to an ergodic random field $\mathbb{Q}$ and, independently, σ is randomly chosen according to $\mathbb{P}$. We denote by $s(\mathbb{Q}|\mathbb{P})$ the relative entropy density of $\mathbb{Q}$ with respect to $\mathbb{P}$, where

$$s(\mathbb{Q}|\mathbb{P}) = \lim_{n\to\infty} \frac{1}{n^d} \sum_{A_n \in \{+,-\}^{\mathcal{C}_n}} \mathbb{Q}(A_n) \log \frac{\mathbb{Q}(A_n)}{\mathbb{P}(A_n)} \cdot$$

We have the following result (proved in Subsection 4.8):

Theorem 7. Assume that $\mathbb{Q}$ is an ergodic random field. Then there exists $\epsilon_0 > 0$ such that for all $\epsilon > \epsilon_0$

$$-\epsilon \log n \leqslant \log \left(\mathbf{W}(\mathcal{C}_n, \eta, \sigma)) \ \mathbb{P}(\eta_{\mathcal{C}_n}) \right) \leqslant \log \log n^{\epsilon} \tag{3.9}$$

for $\mathbb{Q} \otimes \mathbb{P}$-eventually almost every (η, σ). In particular

$$\lim_{n\to\infty} \frac{1}{n^d} \log \mathbf{W}(\mathcal{C}_n, \eta, \sigma) = s(\mathbb{Q}) + s(\mathbb{Q}|\mathbb{P}) \quad \mathbb{Q} \otimes \mathbb{P} - \text{a.s.} \tag{3.10}$$

Remark 3. If in (3.10) we choose $\mathbb{Q} = \mathbb{P}^-$, the low-temperature minus-phase, we conclude that the time to observe a pattern typical for the minus phase in the plus phase, is equal to the time to observe a pattern typical for the plus phase, at the logarithmic scale.

The next theorem is proved in Subsection 4.9.

Theorem 8. For all $q \in \mathbb{R}$ the limit

$$\mathcal{W}(q) = \lim_{n\to\infty} \frac{1}{n^d} \log \int \mathbf{W}(\mathcal{C}_n, \eta, \sigma)^q \; d\mathbb{P}\otimes\mathbb{P}(\eta, \sigma)$$

exists and equals

$$\mathcal{W}(q) = \begin{cases} P((1-q)\beta) + (q-1)P(\beta) & \text{for } q \geqslant -1 \\ P(2\beta) - 2P(\beta) & \text{for } q < -1 \end{cases} \tag{3.11}$$

where P is the pressure defined in (2.2).

From this result, it follows that the sequence $(\frac{1}{n^d} \log \mathbf{W}(\mathcal{C}_n, \eta, \sigma))$ satisfies a generalized large deviation principle in the sense of Theorem 4.5.20 in ref. 8. The differentiability of $q \mapsto P(q\beta)$ would imply a full large deviation principle.

Remark 4. A more general version of Theorem 8 can be easily derived: The measure $\mathbb{P}\otimes\mathbb{P}$ can be replaced by the measure $\mathbb{Q}\otimes\mathbb{P}$ where $\mathbb{Q}$ is any Gibbsian random field (without any mixing assumption). Of course formula (3.11) has to be properly modified (see ref. 4).

For the matching times, we have the following analogue of Theorem 7 (see Subsection 4.10):

Theorem 9. There exists $\epsilon_0 > 0$ such that for all $\epsilon > \epsilon_0$

$$-\epsilon \log n \leqslant \log \left(\mathbf{M}(\mathcal{C}_n, \eta, \sigma) \; \mathbb{P}\otimes\mathbb{P}(\sigma_{\mathcal{C}_n} = \eta_{\mathcal{C}_n})\right) \leqslant \log\log n^{\epsilon} \tag{3.12}$$

for $\mathbb{P}\otimes\mathbb{P}$-eventually almost every (η, σ). In particular

$$\lim_{n\to\infty} \frac{1}{n^d} \log \mathbf{M}(\mathcal{C}_n, \eta, \sigma) = \mathcal{W}(-1) \quad \mathbb{P}\otimes\mathbb{P} - \text{a.s.} \tag{3.13}$$

4. PROOFS

From now on, we write A for A_n to alleviate notations. (Therefore A is understood to be a pattern supported on $\mathcal{C}_n$.)

4.1. Positivity of the Parameter

The following lemma is the analogue of Lemma 4.3 in ref. 4.

Lemma 2 (The parameter). There exist strictly positive constants Λ_1, Λ_2 such that for any integer t with $t\mathbb{P}(A) \leqslant 1/2$, one has

$$\Lambda_1 \leqslant \lambda_{A,t} := -\frac{\log \mathbb{P}(\mathbf{T}_A > t)}{t\mathbb{P}(A)} \leqslant \Lambda_2.$$

Proof. We proceed by estimating the second moment of the random variable $N(A, \mathcal{C}_k, \sigma)$, where $k = \lfloor t^{1/d} \rfloor$. We have

$$\mathbb{E}(N(A, \mathcal{C}_k, \sigma))^2 = \sum_{x,y: x+\mathcal{C}_n \subseteq \mathcal{C}_k, y+\mathcal{C}_n \subseteq \mathcal{C}_k} \mathbb{P}(\theta_x A \cap \theta_y A).$$

We split the sum in three parts: $I_1 = \sum_{x=y}$, $I_2 = \sum_{x \neq y, |x-y| \leqslant \Delta}$, $I_3 = \sum_{x \neq y, |x-y| > \Delta}$, where $\Delta > 0$ will be specified later on.

We now estimate I_1, I_2 and I_3. The quantities I_1 and I_2 are estimated as in ref. 4. For I_1 we have:

$$I_1 = (k+1)^d \mathbb{P}(A).$$

For I_2, using the Gibbs property (2.3) and $d \geqslant 2$:

$$I_2 \leqslant (k+1)^d \Delta^d e^{-\delta n} \mathbb{P}(A).$$

Only the third term involves the disagreement percolation estimate.

$$\begin{aligned} &I_3 - (k+1)^{2d} \mathbb{P}(A)^2 \\ &\leqslant \sum_{x \neq y, |x-y| > \Delta} \mathbb{P}(A) \ |\mathbb{P}(\sigma_{C(x,n)} = A | \sigma_{C(y,n)} = A) - \mathbb{P}(A)|. \end{aligned}$$

Denote by $C'_{x,\Delta,n}$ the set of those sites which are at least at lattice distance $\Delta + 1$ away from $C(x,n)$, and $C^\Delta(x,n)$ the complement of that set. Then we have for $|x-y| > \Delta$:

$$\begin{aligned} &|\mathbb{P}(\sigma_{C(x,n)} = \theta_x A | \sigma_{C(y,n)} = \theta_y A) - \mathbb{P}(A)| \\ &= \left| \iint \left(\mathbb{P}\left(\sigma_{C(x,n)} = \theta_x A | \eta_{C'_{x,\Delta,n}}\right) - \mathbb{P}\left(\sigma_{C(x,n)} = \theta_x A | \xi_{C'_{x,\Delta,n}}\right)\right) d\mathbb{P}(\eta | \sigma_{C(y,n)} = \theta_y A) d\mathbb{P}(\xi) \right| \\ &\leqslant \int\int \mathbb{P}^\eta_{C^\Delta(x,n)} \otimes \mathbb{P}^\xi_{C^\Delta(x,n)} \left(C(x,n) \nleftrightarrow \partial C^\Delta(x,n)\right) d\mathbb{P}(\eta | \sigma_{C(y,n)} = \theta_x A) d\mathbb{P}(\xi) \\ &\leqslant \frac{1}{\mathbb{P}(A)} \ \mathbb{P} \otimes \mathbb{P} \left(C(x,n) \nleftrightarrow \partial C^\Delta(x,n)\right) \leqslant \frac{1}{\mathbb{P}(A)} \ |\partial C(x,n)| \ e^{-d(C(x,n), \partial C^\Delta(x,n))} \\ &\leqslant e^{-cn^{d+1} + c'n^d} \leqslant e^{-\tilde{c} n^{d+1}} \end{aligned}$$

where in the last step we made the choice $\Delta = \Delta_n = n^{d+1}$. Using the second moment estimate (Lemma 4.2 in ref. 4) and proceeding as in the proof of Lemma 4.3 in ref. 4, we obtain the inequality

$$\frac{\mathbb{P}(\mathbf{T}_A \leqslant t)}{t\mathbb{P}(A)} \geqslant \frac{1}{1+e^{-\delta n}\Delta^d + t\mathbb{P}(A) + e^{-cn^{d+1}} t/\mathbb{P}(A)} \geqslant \frac{1}{1+C_1+1/2+C_2}$$

where

$$C_1 = \sup_n n^{d(d+1)}\ e^{-\delta n} < \infty, \quad C_2 = \sup_A \sup_{t\leqslant 1/(2\mathbb{P}(A))} e^{-cn^{d+1}} t/\mathbb{P}(A) < \infty.$$

The upper bound is derived as in the high temperature case, see ref. 4. ∎

4.2. Iteration Lemma and Proof of Theorem 1

This is the analogue of Lemma 4.4 in ref. 4.

We consider k mutually disjoint cubes C_i such that $|C_i| = f_A = (\lfloor \mathbb{P}(A)^{-\theta/d}\rfloor + 1)^d$, where $0 < \theta < 1$ is fixed. The essential point is to make precise the approximation of $\mathbb{P}(A \not\prec \cup_{i=1}^k C_i)$ by $\mathbb{P}(A \not\prec C_1)^k$.

For a cube C_i we denote by $C_i^{\Delta'} \subseteq C_i$ the largest cube inside C_i with the same midpoint as C_i and such that the boundary ∂C_i is at least at lattice distance Δ' away from $C_i^{\Delta'}$, where $\Delta' = \Delta'(n,t) > n^{d+1}$ will be fixed later. We have

$$\begin{aligned}
&\mathbb{P}\left(A \not\prec \cup_{i=1}^k C_i\right)\\
&\quad = \mathbb{P}(A \not\prec C_1 | A \not\prec C_2 \cap A \not\prec C_3 \cap \cdots \cap A \not\prec C_k)\\
&\qquad \times \mathbb{P}(A \not\prec C_2 \cap A \not\prec C_3 \cap \cdots \cap A \not\prec C_k)\\
&\quad = \left(\mathbb{P}(A \not\prec C_1^{\Delta'} | A \not\prec C_2 \cap A \not\prec C_3 \cap \cdots \cap A \not\prec C_k) + \epsilon_1\right)\\
&\qquad \times \mathbb{P}(A \not\prec C_2 \cap A \not\prec C_3 \cap \cdots \cap A \not\prec C_k)\\
&\quad = \left(\mathbb{P}(A \not\prec C_1^{\Delta'}) + \epsilon_1 + \epsilon_2\right) \mathbb{P}(A \not\prec C_2 \cap A \not\prec C_3 \cap \cdots \cap A \not\prec C_k)\\
&\quad = (\mathbb{P}(A \not\prec C_1) + \epsilon_1 + \epsilon_2 + \epsilon_3)\, \mathbb{P}(A \not\prec C_2 \cap A \not\prec C_3 \cap \cdots \cap A \not\prec C_k).
\end{aligned}$$

We now start to estimate the errors ϵ_i. For the first one:

$$\begin{aligned}
|\epsilon_1| &\leqslant \mathbb{P}(A \not\prec C_1^{\Delta'} \cap A \prec C_1 | A \not\prec C_2 \cap A \not\prec C_3 \cap \cdots \cap A \not\prec C_k)\\
&\leqslant \Delta' f_A^{(d-1)/d} \mathbb{P}(A)\ e^{cn^{d-1}}.
\end{aligned}$$

In the last step, the factor $e^{cn^{d-1}}$ arises by removing the conditioning and using the following general property of Gibbs measures:

$$\sup_{\eta,\xi} \frac{\mathbb{P}(\sigma_{\mathcal{C}_n} = A | \eta_{\mathcal{C}_n^c})}{\mathbb{P}(\sigma_{\mathcal{C}_n} = A | \xi_{\mathcal{C}_n^c})} \leqslant e^{cn^{d-1}}.$$

For ϵ_2 we use the disagreement percolation estimate, as in the proof of Lemma 2:

$$|\epsilon_2| \leqslant \frac{\mathbb{P} \otimes \mathbb{P}(C_1^{\Delta'} \not\leftrightarrow \partial C_1)}{\mathbb{P}(A \not\prec C_2 \cap A \not\prec C_3 \cap \cdots \cap A \not\prec C_k)} \leqslant e^{-c_1 \Delta'} e^{c_2 n^d} \leqslant e^{-cn^{d+1}}$$

where $c_1, c_2, c > 0$. Finally, proceeding as in the estimation of ϵ_1, we get

$$\epsilon_3 \leqslant \Delta' f_A^{(d-1)/d} \mathbb{P}(A)$$

where now the boundary factor $e^{cn^{d-1}}$ is absent since we do not have a conditioned measure. Let

$$\alpha_{k-p} = \mathbb{P}(A \prec \cup_{i=p+1}^{k} C_i).$$

We obtain the recursion inequality:

$$\alpha_k \leqslant (\alpha_1 + \epsilon_1 + \epsilon_3)\alpha_{k-1} + \epsilon$$

where $\epsilon \leqslant e^{-cn^{d+1}}$. Following the lines of the proof of Lemma 4.4 in ref. 4 [formula (38)] this gives

$$\begin{aligned} &\alpha_k - \alpha_1^k \leqslant \\ &k\Big(2\Delta' f_A^{(d-1)/d} \mathbb{P}(A) e^{cn^{d-1}}\Big)\Big(\mathbb{P}(A \not\prec C_1) + 2\Delta' f_A^{(d-1)/d} \mathbb{P}(A) e^{cn^{d-1}}\Big)^{k-1} \\ &+ k\epsilon =: \mathrm{I} + \mathrm{II}. \end{aligned}$$

Now, fix $f_A = \mathbb{P}(A)^{-\theta}$, $\Delta' = tn^{d+1}$ and $k = \lfloor \frac{t}{\mathbb{P}(A) f_A} \rfloor$. Then we have

$$\mathrm{I} \leqslant te^{-cn^d}.$$

and

$$\mathrm{II} \leqslant te^{-ctn^{d+1}}.$$

Therefore, as long as $t < e^{\kappa n^d}$ with $\kappa < c$, we have

$$\alpha_k - \alpha_1^k \leqslant e^{-c'n^d} e^{-ct}.$$

The lower bound

$$\alpha_k - \alpha_1^k \geqslant e^{-c'n^d} e^{-ct}$$

is obtained analogously. At this stage, one can repeat the proof of ref. 4 to obtain (3.1) in Theorem 1. ∎

4.3. Return Time

Let $C = \mathcal{C}_{f_A^{1/d}}$ where $f_A = (\lfloor \mathbb{P}(A)^{-\theta/d} \rfloor + 1)^d$. (Notice that $\mathcal{C}_n \subseteq C$ as long as n is large enough.) For a pattern $A = A_n$ and a configuration $\sigma \in \Omega$ such that $\sigma_{\mathcal{C}_n} = A$ we write $A \prec^* C$ for the event that A appears at least twice C and $A \not\prec^* C$ is the event that A occurs in C only on $\mathcal{C}_n$, i.e., the number of occurrences is equal to one.

In order to repeat the iteration lemma for pattern repetitions, we first prove the following lemma.

Lemma 3. Let $A = A_n$ be a good pattern, then there exists $c > 0$ such that for the cube $C = \mathcal{C}_{f_A^{1/d}}$ where $f_A = (\lfloor \mathbb{P}(A)^{-\theta/d} \rfloor + 1)^d$, we have

$$\left|\mathbb{P}(A \not\prec^* C | A) - \mathbb{P}(A \not\prec C)\right| \leqslant e^{-cn^d}.$$

Proof. Since A is good, A does not appear in any cube $\theta_x \mathcal{C}_n$ for $|x| < n/2$. We will introduce a gap Δ with a n-dependence to be chosen later on. Denote by $\mathcal{C}_n^{\Delta}$ the minimal cube containing $\mathcal{C}_n$ such that its boundary is at distance at least Δ from $\mathcal{C}_n$. We have

$$\begin{aligned} |\mathbb{P}(A \not\prec^* C | A) - \mathbb{P}(A \not\prec^* C \setminus \mathcal{C}_n^{\Delta} | A)| &\leqslant \mathbb{P}(A \prec \mathcal{C}_n^{\Delta+n+1} \setminus \mathcal{C}_{n/2} | A) \\ &\leqslant (\Delta + n + 1)^d e^{-cn^d}. \end{aligned}$$

To get the last inequality, remark that

$$\mathbb{P}(A \prec \mathcal{C}_n^{\Delta+n+1} \setminus \mathcal{C}_{n/2} | A) \leqslant |\mathcal{C}_n^{\Delta+n+1} \setminus \mathcal{C}_{n/2}| \sup_{V : |V| > (n/2)^d} \sup_{B \in \Omega_V} \sup_{\eta \in \Omega} \mathbb{P}(B | \eta_{V^c}) \tag{4.1}$$

since $|\theta_x \mathcal{C}_n \setminus \mathcal{C}_n| > (n/2)^d$ for $|x| \geqslant n/2$. The rhs of (4.1) is bounded by e^{-cn^d} by the Gibbs property (2.3) and the fact that a conditioning can at most cost a factor $e^{cn^{d-1}}$. Now we can use the mixing property to obtain

$$|\mathbb{P}(A \not\prec^* C \setminus \mathcal{C}_n^\Delta | A) - \mathbb{P}(A \not\prec C \setminus \mathcal{C}_n^\Delta)| \leqslant e^{-c_1 \Delta} e^{c_2 n^d} f_A^{(d-1)/d}$$

and finally,

$$|\mathbb{P}(A \not\prec C) - \mathbb{P}(A \not\prec C \setminus \mathcal{C}_n^\Delta)| \leqslant \Delta f_A^{(d-1)/d} \mathbb{P}(A)$$

which yields the statement of the lemma by choosing $f_A = (\lfloor \mathbb{P}(A)^{-\theta/d} \rfloor + 1)^d$ and $\Delta = n^{d+1}$. ∎

We can now state the analogue of the iteration lemma for pattern repetitions.

Lemma 4. Let $A = A_n \in \mathcal{G}_n$ be a good pattern. Let C_i, $i = 1, \dots, k$, be a collection of disjoint cubes of volume f_A such that $C_1 = \mathcal{C}_{f_A^{1/d}}$. We have the following estimate:

$$\begin{aligned} &\mathbb{P}(A \not\prec^* \cup_{i=1}^k C_i | A) - [\mathbb{P}(A \not\prec C_1)]^k \\ &\leqslant k\Big(2\Delta f_A^{(d-1)/d} \mathbb{P}(A) e^{cn^{d-1}}\Big)\Big(\mathbb{P}(A \not\prec C_1) + 2\Delta f_A^{(d-1)/d} \mathbb{P}(A) e^{cn^{d-1}}\Big)^{k-1} \\ &+ k e^{-c\Delta} + e^{-cn^d} \mathbb{P}(A \not\prec C_1)^{k-1}. \end{aligned}$$

Proof. Start with the following identity:

$$\mathbb{P}(A \not\prec^* \cup_{i=1}^k C_i | A) = \frac{\mathbb{P}(A \cap A \not\prec^* C_1 \cap A \not\prec C_2 \cap \cdots \cap A \not\prec C_k)}{\mathbb{P}(A)} \cdot \tag{4.2}$$

We can proceed now as in the proof of the iteration lemma to approximate the rhs of (4.2) by

$$\Pi_k = \frac{\mathbb{P}(A \cap A \not\prec^* C_1)}{\mathbb{P}(A)} \; \mathbb{P}(A \not\prec C_2) \cdots \mathbb{P}(A \not\prec C_k)$$

at the cost of an error ϵ which can be estimated by

$$\epsilon \leqslant k\Big(2\Delta f_A^{(d-1)/d} \mathbb{P}(A) e^{cn^{d-1}} \Big(\mathbb{P}(A \not\prec C_1) + 2\Delta f_A^{(d-1)/d} \mathbb{P}(A) e^{cn^{d-1}}\Big)^{k-1} + k e^{-c\Delta}.$$

Now, to replace Π_k by $\mathbb{P}(A \not\prec C_1)^k$, use Lemma 3 to conclude that this replacement induces an extra error which is at most

$$e^{-cn^d}\mathbb{P}(A \not\prec C_1)^{k-1}. \tag{4.3}$$

The lemma is proved. ∎

4.4. Matching Time

In order to prove the exponential law (3) for matching times, we first remark that for cylinders A_n defined on $\Omega\times\Omega=(\{+,-\}\times\{+,-\})^{\mathbb{Z}^d}$, we have the analogue of Theorem 1 under the measure $\mathbb{P}\otimes\mathbb{P}$ with the same proof. Indeed, a typical configuration drawn from $\mathbb{P}\otimes\mathbb{P}$ is a sea of $(+,+)$ with exponentially damped islands of non $(+,+)$. We now generalize the statement of Theorem 1 to the $\mathcal{F}_n$ measurable events that we need (which are not cylindrical).

Lemma 5. Suppose $E_n=\{(\sigma,\eta)\colon \sigma_x=\eta_x,\ \forall x\in\mathcal{C}_n\}$. Theorem 1 holds with A_n replaced by E_n and $\mathbb{P}$ replaced by $\mathbb{P}\otimes\mathbb{P}$.

Proof. Clearly, the analogue of the iteration lemma does not pose any new problem. The main point is to prove the non-triviality of the parameter, i.e., the analogue of Lemma 2. In order to obtain this, we have to estimate the second moment of

$$N^k_{E_n}=\sum_{x:\mathcal{C}_n+x\subseteq C_k} I(\theta_x E_n)$$

under $\mathbb{P}\otimes\mathbb{P}$. As before we split

$$\mathbb{E}\times\mathbb{E}(N^k_{E_n})^2\leqslant I_1+I_2+I_3 \tag{4.4}$$

where $I_1=\sum_{x=y}\mathbb{P}\otimes\mathbb{P}(E_n)\leqslant(k+1)^d\mathbb{P}(E_n)$, $I_2=\sum_{x\neq y,|x-y|\leqslant\Delta}\mathbb{P}\otimes\mathbb{P}(\theta_x E_n\cap\theta_y E_n)$ and $I_3=\sum_{x\neq y,|x-y|>\Delta}\mathbb{P}\otimes\mathbb{P}(\theta_x E_n\cap\theta_y E_n)$. The only problematic term here is I_2. As in the proof for cylindrical events, we will use the Gibbs property, and prove first the existence of $1>\delta>0$ such that

$$\delta\leqslant\mathbb{P}\otimes\mathbb{P}(\sigma_x=\eta_x\,|\,(\sigma,\eta)_{\mathbb{Z}^d\setminus\{x\}})\leqslant 1-\delta. \tag{4.5}$$

We now further estimate

$$\begin{aligned}\mathbb{P}\otimes\mathbb{P}(\sigma_x=\eta_x|(\sigma,\eta)_{\mathbb{Z}^d\setminus\{x\}}) &= \sum_{\epsilon=+,-}\mathbb{P}(\sigma_x=\epsilon|\sigma)\mathbb{P}(\eta_x=\epsilon|\eta)\\ &\leqslant \sup_{\sigma,\eta}[\mathbb{P}(+|\sigma)\mathbb{P}(+|\eta)+(1-\mathbb{P}(+|\sigma))(1-\mathbb{P}(+|\eta))].\end{aligned} \tag{4.6}$$

Since by the Gibbs property $0<\zeta<\mathbb{P}(+|\eta)<1-\zeta<1$, we can bound (4.6) by

$$\max_{\zeta<x,y<1-\zeta}(2uv-u-v-1)<1$$

where the last inequality follows from

$$2uv\leqslant u^2+v^2<u+v$$

for $u,v<1-\zeta<1$. From inequality (4.5), we obtain using $d\geqslant 2$:

$$\begin{aligned}&\sum_{x\in C_k}\sum_{y\neq x,|y-x|\leqslant\Delta}\mathbb{P}\otimes\mathbb{P}(\theta_y E_n|\theta_x E_n)\ \mathbb{P}\otimes\mathbb{P}(E_n)\\ &\leqslant (k+1)^d(\Delta+1)^d\sup_{\sigma,\eta}\sup_{k\geqslant n}\sup_{x_1,\dots,x_k\in\mathbb{Z}^d}\\ &\mathbb{P}\otimes\mathbb{P}(\sigma_{x_1}=\eta_{x_1},\dots,\sigma_{x_k}=\eta_{x_k}|(\sigma,\eta)_{\mathbb{Z}^d\setminus\{x_1,\dots,x_k\}})\\ &\leqslant(1-\delta)^n.\end{aligned}$$

Therefore, choosing $\Delta=n^{d+1}$, we obtain

$$\sum_{x\in C_k}\sum_{y\neq x,|y-x|\leqslant\Delta}\mathbb{P}\otimes\mathbb{P}(\theta_y E_n|\theta_x E_n)\mathbb{P}\otimes\mathbb{P}(E_n)\leqslant(k+1)^d C$$

where

$$C=\sup_n n^{d(d+1)}(1-\delta)^n<\infty.$$

The third term in the decomposition (4.4) is estimated as in the proof of Lemma 2. At this point we can repeat the proof of Lemma 2. ▮

4.5. Poisson Law for Occurrences

For a *good pattern* $A = A_n$ supported on $\mathcal{C}_n$, we define the second occurrence time by the relation:

$$(T_A^2(\sigma) \leqslant k^d) = (N(A, V_k, \sigma) \geqslant 2)$$

and the restriction that T_A^2 can only take values $(k+1)^d$, $k \in \mathbb{N}$. Similarly we define the pth occurrence time:

$$(T_A^p(\sigma) \leqslant k^d) = (N(A, V_k, \sigma) \geqslant p)$$

and the same restriction. The following proposition shows that in the limit $n \to \infty$, properly normalized increments of the process $\{T_{A_n}^k : k \in \mathbb{N}\}$ converge to a sequence of independent exponentials. This implies convergence of the finite dimensional distributions of the counting process to a Poisson process defined in (3.4).

Proposition 1. Let A_n be a good pattern (in the sense of Definition 1). Define $\tau_{A_n}^p = T_{A_n}^p - T_{A_n}^{p-1}$, where $T_{A_n}^0 = 0$. For all $p \in \mathbb{N}$, $t_1, \ldots, t_p \in [0, \infty)$,

$$\lim_{n\to\infty} \mathbb{P}\left(\left[\tau_{A_n}^p \geqslant \sum_{i=1}^{p} t_i/\mathbb{P}(A_n)\right] \cap \left[\tau_{A_n}^{p-1} \leqslant \sum_{i=1}^{p-1} t_i/\mathbb{P}(A_n)\right] \cap \cdots \cap \left[\tau_{A_n}^1 \leqslant t_1/\mathbb{P}(A_n)\right]\right)$$
$$= e^{-(t_1+\cdots t_k)}(1 - e^{-(t_1+\cdots t_{k-1})}) \cdots (1 - e^{-t_1}).$$

Proof. We start with the case of two occurrence times T_1, T_2:

$$\mathbb{P}\left(T_1 \leqslant \frac{t}{\mathbb{P}(A)} \cap T_2 \geqslant \frac{s}{\mathbb{P}(A)} + T_1\right)$$
$$= \sum_{k \leqslant \frac{t}{\mathbb{P}(A)}} \mathbb{P}\left(T_2 \geqslant \frac{s}{\mathbb{P}(A)} + k \;\middle|\; T_1 = k\right) \mathbb{P}(T_1 = k).$$

Let us denote by $\mathcal{C}_k$ the cube defined by the relation $(T_1 \leqslant k) = (A \prec \mathcal{C}_k)$, and by $A \prec^1 C_k$ the event that A appears for the first time in C_k (more precisely $A \prec^1 C_k$ abbreviates the event $(T_1 = k)$, i.e., $\cap_{l<k}(A \not\prec C_l) \cap (A \prec C_k)$).

Let us denote by $\mathcal{C}_k^\Delta$ the Δ-extension of $\mathcal{C}_k$, i.e., the minimal cube containing $\mathcal{C}_k$ such that $\partial\mathcal{C}_k^\Delta$ and ∂C_k are at least Δ apart. Recall that

$C(t/\mathbb{P}(A))$ denotes the maximal cube of the form $C_k=[0,k]^d\cap\mathbb{Z}^d$ such that $|C_k|\leqslant t/\mathbb{P}(A)$. Remember that

$$\frac{|C(t/\mathbb{P}(A))|}{t/\mathbb{P}(A)}\to 1$$

as $n\to\infty$.

Lemma 6. If A is a good pattern, then we have the estimate

$$\mathbb{P}\left(T_2\geqslant\frac{s}{\mathbb{P}(A)}+k\;\Big|\;A\prec^1\mathcal{C}_k\right)-\mathbb{P}\Big(A\not\prec C\left(\frac{s}{\mathbb{P}(A)}\right)\setminus\mathcal{C}_k^\Delta\;\Big|\;A\prec^1\mathcal{C}_k\Big)$$
$$\leqslant\Delta f_A^{(d-1)/d}e^{-cn^d}.$$

Proof. The proof is identical to that of Lemma 3. ∎

Now we want to replace

$$\mathbb{P}\Big(A\not\prec C\left(\frac{s}{\mathbb{P}(A)}\right)\setminus\mathcal{C}_k^\Delta\;\Big|\;A\prec^1\mathcal{C}_k\Big)$$

by the unconditioned probability of the same event. We make the choice $\Delta=n^{d+1}$. By the disagreement percolation estimate, this gives an error which can be bounded by

$$\sum_{k\leqslant t/\mathbb{P}(A)}\mathbb{P}(T_1=k)\Big[\mathbb{P}\Big(A\not\prec C\left(\tfrac{s+t}{\mathbb{P}(A)}\right)\setminus\mathcal{C}_k^\Delta\Big|A\prec^1\mathcal{C}_k\Big)-\mathbb{P}\Big(A\not\prec C\left(\tfrac{s+t}{\mathbb{P}(A)}\right)\setminus\mathcal{C}_k^\Delta\Big)\Big]$$
$$\leqslant\sum_{k\leqslant t/\mathbb{P}(A)}e^{-c\Delta}\leqslant t^2e^{cn^d}e^{-c'n^{d+1}}.$$

Finally, we have

$$\sup_{k\leqslant t/\mathbb{P}(A)}\Big[\mathbb{P}\Big(A\not\prec C\left(\frac{s+t}{\mathbb{P}(A)}\right)\setminus\mathcal{C}_k^\Delta\Big)-\mathbb{P}\Big(A\not\prec C\left(\frac{s+t}{\mathbb{P}(A)}\right)\setminus C\left(\frac{t}{\mathbb{P}(A)}\right)\Big)\Big]$$
$$\leqslant\Delta(t/\mathbb{P}(A))^{(d-1)/d}\mathbb{P}(A)=\Delta t^{(d-1)/d}\mathbb{P}(A)^{1/d}.$$

By the exponential law, we have, using $|C((t+s)/\mathbb{P}(A))\setminus C(t/\mathbb{P}(A))|=t/\mathbb{P}(A)$:

$$\mathbb{P}\left(A\not\prec C\left(\frac{s+t}{\mathbb{P}(A)}\right)\setminus C\left(\frac{t}{\mathbb{P}(A)}\right)\right)=\exp(-\lambda_A s)+\epsilon_n$$

where $\epsilon_n = \epsilon(n, t, s) \to 0$ as $n \to \infty$. Which gives:

$$\lim_{n\to\infty} \left(\mathbb{P}(\tau_2 \geqslant s/\mathbb{P}(A) \cap \tau_1 \leqslant t/\mathbb{P}(A)) - \lim_n \mathbb{P}(\tau_1 \leqslant t/\mathbb{P}(A))e^{-\lambda_A s}\right)$$
$$= \lim_{n\to\infty} \left(\mathbb{P}(\tau_2 \geqslant s/\mathbb{P}(A) \cap \tau_1 \leqslant t/\mathbb{P}(A)) - (1-e^{-\lambda_A t})e^{-\lambda_A s}\right) = 0.$$

This proves the statement of the proposition for $k=2$, the general case is analogous and left to the reader. ∎

The following proposition follows immediately from Proposition 1.

Proposition 2. Let $A_n \in \mathcal{G}_n$ be a good pattern supported on $\mathcal{C}_n$. Then the finite dimensional marginals of the process $\{N^n_{t/\lambda_{A_n}}: t \geqslant 0\}$ converge to the finite dimensional marginals of a mean one Poisson process as n tends to infinity.

In order to obtain convergence in the Skorokhod space, we have to prove tightness. This is an immediate consequence of the following simple lemma for general point processes, applied to

$$N^n_t = N(A_n, C(t/\mathbb{P}(A_n)), \sigma).$$

Lemma 7. Let $\{N^n_t: t \geqslant 0\}$ be a sequence of point processes with path space measures $\mathbb{P}^T_n$ on $D([0,T],\mathbb{N})$. If there exists $C > 0$ such that for all n and for all $t \leqslant T$ we have the estimate

$$\mathbb{E}^T_n(N^n_t) \leqslant Ct \tag{4.7}$$

then the sequence $\mathbb{P}^T_n$ is tight.

Proof. From (4.7) we infer for all n, $t \leqslant T$

$$\mathbb{P}^T_n(N^n_t \geqslant K) \leqslant CT/K.$$

Hence

$$\lim_{K\uparrow\infty} \sup_{0\leqslant t\leqslant T} \sup_n \mathbb{P}^T_n(N^n_t \geqslant K) = 0 \tag{4.8}$$

For a trajectory $\omega \in D([0,T],\mathbb{N})$ one defines the modulus of continuity

$$w_\gamma(T,\omega) = \inf_{(t_i)_{i=1}^N} \sup_{i=1}^N |\omega_{t_i} - \omega_{t_{i-1}}|$$

where the infimum is taken over all partitions $t_0 = 0 < t_1 < \cdots < t_N = t$ such that $t_i - t_{i-1} \geqslant \gamma$. If for some $\epsilon > 0$ $w_\gamma(T, \omega) \geqslant \epsilon$, then the number of jumps of ω in $[0, T]$ is at least $[T/\gamma]$. Hence we obtain using (4.7):

$$\mathbb{P}_n^T(w_\gamma(T, \omega) \geqslant \epsilon) \leqslant \mathbb{P}_n^T(N_T^n \geqslant T/\gamma) \leqslant C\gamma.$$

This gives for all $\epsilon > 0$:

$$\lim_{\gamma \downarrow 0} \sup_n \mathbb{P}_n^T(w_\gamma(T, \omega) \geqslant \epsilon) = 0. \tag{4.9}$$

Combination of (4.8) and (4.9) with the tightness criterion (ref. 14, p. 152) yields the result. ▮

Remark 5. With much more effort, one can obtain precise bounds for the difference

$$\left|\mathbb{P}(N_t^n/\lambda_{A_n} = k) - \frac{t^k}{k!} e^{-t}\right|$$

which are well-behaved in n, t and k. In particular, from such bounds one can obtain convergence of all moments of N_t^n/λ_{A_n} to the corresponding Poisson moments. This is done in ref. 1 in the context of mixing processes.

4.6. Gumbel Law

For $\eta, \sigma \in \Omega$ denote

$$\mathcal{V}_0(\eta, \sigma) = \bigcup \{G_k : \sigma_{G_k} = \eta_{G_k}\}.$$

We start with the following simple lemma:

Lemma 8. 1. There exists $\delta > 0$ such that for all $\eta \in \Omega$:

$$\inf_{k \in \mathbb{N}} \frac{\mathbb{P} \otimes \mathbb{P}(\mathcal{V}_0 \supset G_{k+1})}{\mathbb{P}(\mathcal{V}_0 \supset G_k)} \geqslant \delta.$$

2. There exists a non-decreasing sequence $u_n \uparrow \infty$ such that for all $n \in \mathbb{N}$:

$$1 \leqslant n\mathbb{P}(\mathcal{V}_0 \supset G_{u_n}) \leqslant \frac{1}{\delta} \cdot$$

Proof. For item 1:

$$\begin{aligned}\frac{\mathbb{P}\otimes\mathbb{P}(\mathcal{V}_0\supset G_{k+1})}{\mathbb{P}\otimes\mathbb{P}(\mathcal{V}_0\supset G_k)} &= \mathbb{P}\otimes\mathbb{P}(\eta_{x_{n+1}}=\sigma_{x_{n+1}}|\sigma_{G_n}=\eta_{G_n})\\ &\geqslant \inf_{\xi,\sigma}\mathbb{P}\otimes\mathbb{P}(\sigma_x=\eta_x|\sigma_{\mathbb{Z}^d\setminus\{x\}},\xi_{\mathbb{Z}^d\setminus\{x\}})\\ &=\delta>0\end{aligned}$$

where the last inequality follows from the fact that $\mathbb{P}\otimes\mathbb{P}$ is a Gibbs measures. For item 2, put

$$f(n)=\mathbb{P}\otimes\mathbb{P}(\mathcal{V}_0\supset G_n)$$

and

$$\begin{aligned}u_n^+ &= \min\{k\colon f(k)\leqslant 1/n\}\\ u_n^- &= \max\{k\colon f(k)\geqslant 1/n\}\end{aligned}$$

Clearly,

$$u_n^-\leqslant u_n^+\leqslant u_n^-+1.$$

Now choosing $u_n=u_n^-$ and using (4.10), we obtain

$$\begin{aligned}\frac{1}{n}&\leqslant\mathbb{P}\otimes\mathbb{P}(\mathcal{V}_0\supset G_{u_n})\\ &=\frac{\mathbb{P}\otimes\mathbb{P}(\mathcal{V}_0\supset G_{u_n})}{\mathbb{P}\otimes\mathbb{P}(\mathcal{V}_0\supset G_{u_n+1})}\ \mathbb{P}\otimes\mathbb{P}(\mathcal{V}_0\supset G_{u_n+1})\\ &\leqslant\frac{1}{\delta n}\cdot\quad\blacksquare\end{aligned}$$

We now adapt our definition of matching time to the sequence of sets G_n:

$$\tau_n^{\mathcal{G}}(\eta,\sigma)=\min\{k\colon \exists x\colon G_n+x\subseteq G_k \text{ such that } \sigma_{G_n+x}=\eta_{G_n+x}\}.$$

We have the relation

$$(\mathcal{M}_n(\eta,\sigma)\geqslant k)=(\tau_k^{\mathcal{G}}(\eta,\sigma)\leqslant n).$$

In words: the maximal matching inside G_n is greater than or equal to k if and only if the first time that a matching on a set G_k happens is not larger than n. Now we choose $k = u_n + x$ ($x \in \mathbb{N}$) and use the exponential law for matching times:

$$\mathbb{P} \otimes \mathbb{P}(\tau^{\mathcal{G}}_{u_n}(\eta, \sigma) \leqslant n) = 1 - \exp(-\lambda_n \mathbb{P} \otimes \mathbb{P}(\sigma_{G_{u_n+x}} = \eta_{G_{u_n+x}})) + \epsilon_n$$

where ϵ_n goes to zero as n goes to infinity. By the choice of u_n,

$$\mathbb{P} \otimes \mathbb{P}(\sigma_{G_{u_n+x}} = \eta_{G_{u_n+x}}) = \mathbb{P} \otimes \mathbb{P}(\mathcal{V}_0(\eta, \sigma) \supset G_{u_n+x}) \in \left[\frac{A}{n} e^{-\nu x}, \frac{B}{n} e^{-\nu' x} \right] \tag{4.10}$$

where $A, B \in (0, \infty)$ and

$$0 < e^{-\nu} = \liminf_{n \to \infty} \frac{\mathbb{P} \otimes \mathbb{P}(\sigma_{G_{n+1}} = \eta_{G_{n+1}})}{\mathbb{P} \otimes \mathbb{P}(\sigma_{G_n} = \eta_{G_n})} < 1$$

and

$$0 < e^{-\nu'} = \limsup_{n \to \infty} \frac{\mathbb{P} \otimes \mathbb{P}(\sigma_{G_{n+1}} = \eta_{G_{n+1}})}{\mathbb{P} \otimes \mathbb{P}(\sigma_{G_n} = \eta_{G_n})} < 1.$$

Here the inequality for the $\liminf$ is an immediate consequence of Lemma 8, and the inequality for the $\limsup$ is derived in a completely analogous way, using the Gibbs property. The theorem now follows immediately from (4.10).

4.7. Proof of Theorem 6

We start by showing the following summable upper-bound of

$$\begin{aligned} &\mathbb{P}\{\sigma\colon \log(\mathbf{R}_{\sigma_{\mathcal{C}_n}}(\sigma)\mathbb{P}(\sigma_{\mathcal{C}_n})) \geqslant \log t\} \\ &\quad \leqslant \sum_{A_n \in \mathcal{G}_n} \mathbb{P}(A_n)\mathbb{P}\{\sigma\colon \log(\mathbf{R}_{A_n}(\sigma)\mathbb{P}(A_n)) \geqslant \log t \mid A_n\} + \sum_{A_n \in \mathcal{G}_n^c} \mathbb{P}(A_n). \end{aligned}$$

From Theorem 2 and Lemma 1 we get for all $0 < t < e^{\kappa n^d}$

$$\mathbb{P}\{\sigma\colon \log(\mathbf{R}_{\sigma_{\mathcal{C}_n}}(\sigma)\mathbb{P}(\sigma_{\mathcal{C}_n})) \geqslant \log t\} \leqslant e^{-c' n^d} + e^{-\Lambda_1 t} + e^{-c n^d}.$$

Take $t = t_n = \log(n^{\epsilon})$, $\epsilon > \Lambda_1^{-1}$, to get

$$\mathbb{P}\{\sigma\colon \log(\mathbf{R}_{\sigma_{\mathcal{C}_n}}(\sigma)\mathbb{P}(\sigma_{\mathcal{C}_n})) \geqslant \log\log(n^{\epsilon})\} \leqslant e^{-c'n^d} + \frac{1}{n^{\epsilon\Lambda_1}} + e^{-cn^d}.$$

An application of the Borel–Cantelli lemma leads to

$$\log\left(\mathbf{R}_{\sigma_{\mathcal{C}_n}}(\sigma)\mathbb{P}(\sigma_{\mathcal{C}_n})\right) \leqslant \log\log(n^{\epsilon}) \quad \text{eventually a.s..}$$

For the lower bound first observe that Theorem 2 gives, for all $0 < t < e^{cn^d}$

$$\mathbb{P}\{\sigma\colon \log(\mathbf{R}_{\sigma_{\mathcal{C}_n}}(\sigma)\mathbb{P}(\sigma_{\mathcal{C}_n})) \leqslant \log t\} \leqslant e^{-c'n^d} + 1 - \exp(-\Lambda_2 t) + e^{-cn^d}.$$

Choose $t = t_n = n^{-\epsilon}$, $\epsilon > 1$, to get, proceeding as before,

$$\log\left(\mathbf{R}_{\sigma_{\mathcal{C}_n}}(\sigma)\mathbb{P}(\sigma_{\mathcal{C}_n})\right) \geqslant -\epsilon \log n \quad \text{eventually a.s..}$$

Finally, let $\epsilon_0 = \max(\Lambda_1^{-1}, 1)$.

4.8. Proof of Theorem 7

We first show that the strong approximation formula (3.7) holds with $\mathbf{W}(\mathcal{C}_n, \eta, \sigma)$ in place of $\mathbf{R}_{\sigma_{\mathcal{C}_n}}(\sigma)$ with respect to the measure $\mathbb{Q} \otimes \mathbb{P}$. We have the following identity:

$$\begin{aligned}\int d\mathbb{Q}(\eta)\ \mathbb{P}&\left\{\sigma\colon \mathbf{T}_{\eta_{\mathcal{C}_n}}(\sigma) > \frac{t}{\mathbb{P}(\eta_{\mathcal{C}_n})}\right\}\\ &= (\mathbb{Q}\otimes\mathbb{P})\left\{(\eta,\sigma)\colon \mathbf{W}(\mathcal{C}_n,\eta,\sigma) > \frac{t}{\mathbb{P}(\eta_{\mathcal{C}_n})}\right\}.\end{aligned}$$

This shows that Theorem 1 remains valid if we replace $\mathbf{T}_{\eta_{\mathcal{C}_n}}(\sigma)$ with $\mathbf{W}(\mathcal{C}_n, \eta, \sigma)$ and $\mathbb{P}$ with $\mathbb{Q} \otimes \mathbb{P}$, hence so is Theorem 6. Therefore for ϵ large enough, we obtain

$$-\epsilon \log n \leqslant \log(\mathbf{W}(\mathcal{C}_n, \eta, \sigma)\mathbb{P}(\eta_{\mathcal{C}_n})) \leqslant \log\log n^{\epsilon} \tag{4.11}$$

for $\mathbb{Q}\otimes\mathbb{P}$-eventually almost every (η, σ). Write

$$\log(\mathbf{W}(\mathcal{C}_n, \eta, \sigma)\mathbb{P}(\sigma_{\mathcal{C}_n})) = \log \mathbf{W}(\mathcal{C}_n, \eta, \sigma) + \log \mathbb{Q}(\eta_{\mathcal{C}_n}) - \log \frac{\mathbb{Q}(\eta_{\mathcal{C}_n})}{\mathbb{P}(\eta_{\mathcal{C}_n})}$$

and use (4.11). After division by n^d, we obtain (3.10) since $\lim_{n\to\infty}\frac{1}{n^d}\log\mathbb{Q}(\sigma_{\mathcal{C}_n})=-s(\mathbb{Q})$, $\mathbb{Q}$-a.s. by the Shannon–McMillan–Breiman Theorem and $\lim_{n\to\infty}\frac{1}{n^d}\log\frac{\mathbb{Q}(\eta_{\mathcal{C}_n})}{\mathbb{P}(\eta_{\mathcal{C}_n})}=s(\mathbb{Q}|\mathbb{P})$, $\mathbb{Q}$-a.s. by the Gibbs variational principle (See e.g. ref. 4 for a proof).

4.9. Proof of Theorem 8

We follow the line of proof of ref. 4 to compute $\mathcal{W}(q)$. The only extra complication in our case is that the bound

$$\mathbb{P}\left(\mathbf{T}_{A_n}>\frac{t}{\mathbb{P}(A_n)}\right)\leqslant e^{-ct}$$

for all $t>0$ cannot be obtained directly from Theorem 1. Instead we will use the following lemma which shows that such a bound can be obtained by a rough version of the iteration lemma. Given this result, the proof of ref. 4 can be repeated.

Lemma 9. 1. There exists $c>0$ such that for all patterns $A_n\in\{+,-\}^{\mathcal{C}_n}$

$$\mathbb{P}\left(\mathbf{T}_{A_n}>\frac{t}{\mathbb{P}(A_n)}\right)\leqslant e^{-ct}.$$

2. There exists $\delta\in(0,\frac{1}{2})$ such that for all n and all pattern $A=A_n$

$$0<\delta<\mathbb{P}(T_A>\frac{1}{2\mathbb{P}(A)})<1-\delta<1.$$

Proof. To prove the first inequality, we fill part of the cube $C(t/\mathbb{P}(A))$ with little cubes of size f_A (where f_A is defined in Lemma 4.2), with $k\geqslant t/(2\mathbb{P}(A)f_A)$. The gaps Δ separating the different cubes are taken equal to $\lceil tn^{d+1}\rceil$. We then have the following

$$\mathbb{P}(T_A>t/\mathbb{P}(A))\leqslant\mathbb{P}(A\not\prec\cup_{i=1}^{K}C_i).$$

Notice that we do not have to estimate here the probability that the pattern is not in the gaps since we only need an upper bound. Now

$$\begin{aligned}\alpha_K&=\mathbb{P}(A\not\prec\cup_{i=1}^{K}C_i)\\&=\mathbb{P}(A\not\prec C_1|A\not\prec\cup_{i=2}^{K}C_i)\mathbb{P}(A\not\prec\cup_{i=2}^{K}C_i)=\mathbb{P}(A\not\prec C_1|A\not\prec\cup_{i=2}^{K}C_i)\alpha_{K-1}.\end{aligned}$$

Using the disagreement percolation estimate, we have

$$\mathbb{P}(A \not\to C_1 | A \not\to \cup_{i=2}^{K} C_i) - \alpha_1 \leqslant e^{-\Delta}.$$

Therefore

$$\alpha_K \leqslant \alpha_{K-1}\alpha_1 + e^{-\Delta}.$$

Iterating this inequality gives, using $\Delta = \lceil tn^{d+1} \rceil$,

$$\alpha_K \leqslant \alpha_1^K + e^{-tn^{d+1}} e^{cn^d} t$$

Now we use $K > t/2\mathbb{P}(A) f_A$, and Lemma 2 to obtain:

$$\alpha_K \leqslant (1 - \Lambda_1 f_A \mathbb{P}(A))^{t/(2f_A\mathbb{P}(A))} + e^{-ct}$$

which implies the first inequality of the lemma.

The second inequality follows directly from Lemma 2. ∎

4.10. Proof of Theorem 9

The proof of (3.12) is identical to the proof of (3.9) but using the exponential law for the matching time. Formula (3.13) follows from

$$\mathbb{P}\otimes\mathbb{P}(\sigma_{\mathcal{C}_n} = \eta_{\mathcal{C}_n}) = \sum_{\sigma_{\mathcal{C}_n} \in \{+,-\}^{\mathcal{C}_n}} \mathbb{P}(\sigma_{\mathcal{C}_n})^2$$

and the definition of $\mathcal{W}$.

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